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**ANALYSIS OF TURBULENT LIQUID-METAL  
HEAT TRANSFER IN CHANNELS WITH  
HEAT SOURCES IN THE  
FLUID - POWER-LAW VELOCITY PROFILE**

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# ANALYSIS OF TURBULENT LIQUID-METAL HEAT TRANSFER IN CHANNELS WITH HEAT SOURCES IN THE FLUID - POWER-LAW VELOCITY PROFILE

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## SUMMARY

An analysis is made to determine the heat-transfer characteristics for turbulent flow of a heat-generating liquid metal between parallel plates with wall heat transfer. The internal heat generation is uniform over the channel cross section and along its length. The wall heat transfer is also uniform along the channel length. The analysis applies in the thermal entrance region of the channel as well as far downstream. The fluid is assumed to have a fully developed, turbulent power-law velocity profile which is unchanging throughout the length of the channel. The idealized eddy diffusivity profile proposed by Poppendiek is used.

The solutions depend on the power-velocity exponent  $m$  and diffusivity parameter  $k = \sqrt{1 + 0.01 \bar{\psi} \text{PrRe}^{0.9}}$ , where  $\text{Pr}$  is Prandtl number,  $\text{Re}$  is Reynolds number, and  $\bar{\psi}$  is defined as the average effective value of the ratio of the eddy diffusivity of heat transfer to that for momentum transfer. Numerical results for the wall temperature distribution and Nusselt number variation are presented in graphical form for a  $1/7$ -power-law velocity profile and for values of  $k$  ranging from 1 to 5.

Results for the fully developed Nusselt numbers for liquid metal flow without internal heat generation are compared with existing calculations (based on the assumption that the eddy diffusivities of heat and momentum are equal) and exhibit good agreement.

## INTRODUCTION

The study of flowing liquid metals with volumetric internal heat generation is of current interest in several sectors of modern technology. Such flows may occur, for example, in liquid metal magnetohydrodynamic generators for the generation of electric power in space and in liquid-metal-fueled nuclear reactors. In such devices, the liquid metal will be internally heated by the flowing electric current, by radioactive fission

products, and, perhaps, by viscous dissipation. The fluid flow in these devices may, in addition, be accompanied by wall heat transfer.

This investigation is concerned with fully developed turbulent channel flows in which the internal heat generation per unit volume of liquid is uniform. A parallel-plate channel, which approximates a large-aspect-ratio channel, is analyzed in this work as it has applications in the aforementioned devices.

Attention is focused here on the case where the wall heat flux is uniform along the length of the channel. A factor of importance for the proper operation of these devices is that of maintaining a satisfactory temperature distribution along the channel walls. The designer, therefore, must be able to compute the wall temperature under the conditions of internal heat generation and wall heat transfer. The problem involves studying the effect of internal heat generation and wall heat transfer on the wall temperature distribution.

The heat-transfer behavior for this type of situation has been considered in two previous papers. In reference 1, the effect of an internal heat source on heat transfer in round tubes and flat ducts is considered under the conditions of a uniform velocity profile (or slug flow) and turbulent heat transfer occurring solely by molecular conduction. Results were obtained in both the thermal entrance and fully developed regions.

The solution in reference 2 is for liquid-metal flow in a parallel-plate channel. The limitation to heat transfer by molecular conduction only is eliminated, and the effect of transverse thermal eddy diffusivity variations is included through consideration of a simplified eddy diffusivity profile. Results were obtained in both the thermal entrance and fully developed regions. The established turbulent velocity profile was again approximated by a slug-flow profile.

In the present analysis, this last limitation of uniform velocity profile, or slug flow, is also eliminated. The established turbulent velocity profile is represented by a power-law expression. Together with a parameter (to be presented later) that characterizes the effect of eddy transfer at moderate Prandtl or high Reynolds numbers, the wall temperature distribution and heat-transfer characteristics can be obtained for any value of the exponent in the power-law velocity expression. A value of the exponent of  $1/7$  is used in the evaluation of the solutions.

## ANALYSIS

A necessary prerequisite for the solution of the turbulent convective heat-transfer problem is a specification of the velocity profiles and the heat transfer eddy diffusivity profiles. Attention, therefore, will first be directed to the velocity and eddy diffusivity problems.

## Velocity Distribution

The simplification provided by the slug-flow assumption in references 1 and 2 made it possible to obtain exact mathematical solutions to the governing energy equation. The actual turbulent velocity profile in channel flow is not uniform but, instead, falls off toward zero near the walls. In an attempt to simplify analyses and, yet, deal realistically with turbulent flows confined in pipes and channels, much attention has been given to power laws (ref. 3). In order to approximate in a simple manner the velocity distribution for turbulent viscous flow in a pipe, a power-law velocity distribution of the type

$$u = A \left( \frac{z}{r_o} \right)^m \quad (1)$$

is frequently considered. The exponent  $m$  in this power-law velocity expression has been experimentally determined to be  $1/7$  to a Reynolds number of approximately  $10^5$ . For Reynolds numbers larger than  $10^5$ , a better approximation is obtained by the power one-eighth, one-ninth, or even one-tenth. Therefore, the slug-flow velocity profile is expected to be a good approximation for the actual turbulent flow at very high Reynolds numbers for which the velocity distribution is extremely flat.

Although the power-law velocity expression was developed for flow in circular pipes, subsequent study revealed that it represents satisfactorily the established turbulent velocity distribution for flow between parallel plates (ref. 3) when written in the form (fig. 1)

$$u = A \left( 1 - \frac{y}{a} \right)^m = A(1 - y')^m \quad (2)$$

The mean fluid velocity  $\bar{u}$  of the parallel-plates system can be expressed in terms of the

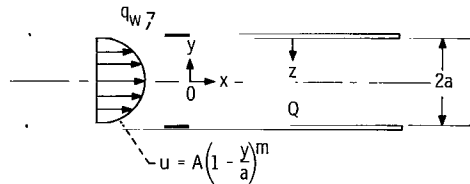


Figure 1. - Geometry and coordinate system. Turbulent velocity profile represented by power-law distribution.

hydrodynamic parameter  $A$  as follows: The mean velocity is given by the definition

$$\bar{u} = \int_0^1 u(y') dy' \quad (3)$$

Substituting equation (2) into equation (3) and carrying out the integration yield

$$A = \bar{u}(m + 1)$$

Therefore, equation (2) can be rewritten as

$$U(y') \equiv \frac{u(y')}{\bar{u}} = (m + 1)(1 - y')^m \quad (4)$$

With an exponent  $m$  of  $1/7$ , equation (4) yields

$$U(y') = \frac{8}{7} (1 - y')^{1/7} \quad (5)$$

while an exponent of zero in equation (4) yields the uniform velocity profile

$$U(y') = 1 \quad (6)$$

Both the  $1/7$ -power-law velocity and the uniform-velocity profiles have been used to gain an understanding of turbulent heat-transfer characteristics for liquid-metal flow in tubes and channels in the absence of internal heat generation (refs. 4 to 11). Both profiles have,

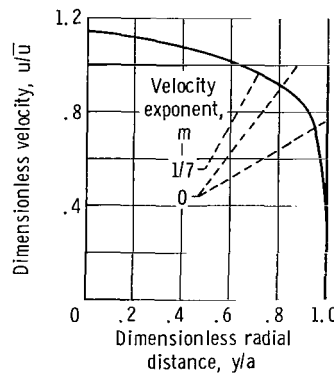


Figure 2. - Velocity profile for fully developed turbulent flow between parallel plates.

in addition, been used in the analyses of hydromagnetic effects in turbulent channel flows (refs. 12 to 15). The assumption of a uniform velocity distribution, for example, is believed reasonably satisfied for a large magnetic-field strength; in practical design calculations, the nonuniformity of the velocity profile would need to be considered when estimating the heating of the fluid caused by ohmic and viscous losses. The velocity profiles represented by equations (5) and (6) are shown in figure 2.

## Eddy Diffusivity Distribution

In most of the previously mentioned analytical studies of turbulent liquid-metal heat transfer, it is postulated that the thermal eddy diffusivity is small compared with the thermal molecular diffusivity and may be neglected; therefore, radial or transverse heat transfer is by molecular conduction only. This is believed a reasonable postulate when the product of Reynolds and Prandtl numbers is below a specified limit (about 100). The practical design of moderate Prandtl or high Reynolds number liquid-metal systems, however, where the effect of radial or transverse eddy diffusivity variations must be included, has prompted the development of idealized eddy diffusivity functions. Poppendiek (refs. 9 and 10) has analyzed turbulent heat transfer for liquid-metal flow between parallel plates for the uniform wall-temperature case and with no internal heat generation by considering a simplified eddy diffusivity function. This function was also used in reference 2. The solution is shown in references 2, 9, and 10 to reduce correctly to known specific solutions of the general case.

The idealized eddy diffusivity profile proposed by Poppendiek (refs. 9 and 10) is used in the present analysis to account for transverse heat flow consisting of eddy transfer as well as molecular conduction. This simplified profile approximates the actual one in the regions nearest the walls over a Reynolds number range of approximately  $5 \times 10^3$  to  $1 \times 10^6$ . For symmetrical wall heating conditions considered herein, the use of this idealization should introduce little error.

The idealized momentum eddy diffusivity distribution varies linearly with distance from the channel centerline and as the nine-tenths power of the Reynolds number:

$$\frac{\epsilon_M}{\nu} = 0.01 \text{ Re}^{0.9} (1 - y') \quad (7)$$

In anticipation of a later need, it will be convenient to introduce the thermal eddy diffusivity parameter  $\lambda_T$  defined by

$$\lambda_T \equiv \text{Pr} \frac{\epsilon_H}{\nu} = 0.01 \psi \text{PrRe}^{0.9} (1 - y') \quad (8a)$$

where  $\psi$  is the ratio of the eddy diffusivity of heat to that of momentum. No generally accepted relation between the two quantities has yet been established. It has been found convenient, however, to consider the parameter  $\bar{\psi}$ , which is the effective average value of  $\psi$  across the channel (ref. 16). The parameter  $\lambda_T$  is then given by

$$\lambda_T = 0.01 \bar{\psi} \text{PrRe}^{0.9} (1 - y') \quad (8b)$$

It is convenient to write the parameter  $\lambda_T$  in terms of a diffusivity parameter  $k$  defined by

$$k^2 \equiv 1 + 0.01 \bar{\psi} \text{PrRe}^{0.9} \quad (9)$$

The parameter  $\lambda_T$  then takes the form

$$\lambda_T = (k^2 - 1)(1 - y') \quad (10)$$

It is noted that heat transport in the fluid occurs solely by molecular conduction (i. e.,  $\lambda_T = 0$ ) for  $y' = 1$  (i. e., at the channel walls) or for  $k = 1$ . It is clear from equation (9) that  $k$  can never physically assume a value less than one.

Now that the velocity and eddy diffusivity distributions have been specified, the solution of the heat-transfer problem is undertaken.

## Heat-Transfer Problem

The geometry of the parallel-plate channel and coordinates are shown in figure 1. The fluid possesses a fully developed velocity profile which is unchanging with length. The fluid enters the channel at  $x = 0$  with a uniform temperature at the value  $t_1$  and is heated both by the internal heat generation in the fluid and by the heat flux at the channel walls. An analysis is performed to determine the resulting wall temperature and heat-transfer characteristics as functions of the axial position. The liquid metal is assumed to have constant physical properties, and only steady-state flow and heat transfer are considered.

The energy equation for the fluid temperature is



$$u \frac{\partial t}{\partial x} = \frac{\partial}{\partial y} \left[ (\alpha + \epsilon_H) \frac{\partial t}{\partial y} \right] + \frac{Q}{\rho c_p} \quad (11)$$

where  $\alpha$  and  $\epsilon_H$  represent, respectively, the molecular and eddy diffusivities for heat. To obtain the energy equation in this form, it has been assumed that viscous dissipation and axial heat conduction are both negligible compared with heat conduction in the transverse direction. This assumption has been shown (ref. 17) to introduce a negligible error for  $Pe > 100$ .

It is convenient to write the temperature  $t$  as the sum of the temperature for the problem where there is a uniform volumetric internal heat generation  $Q$  in an insulated channel  $t_Q$ , and the temperature for the problem where there is a wall heat transfer  $q_w$  without internal heat generation  $t_q$ . Then the temperature in the combined problem is simply given by

$$t = t_Q + t_q \quad (12)$$

The governing equations and boundary conditions for  $t_Q$  and  $t_q$  may be written as

$$u \frac{\partial t_Q}{\partial x} = \frac{\partial}{\partial y} \left[ (\alpha + \epsilon_H) \frac{\partial t_Q}{\partial y} \right] + \frac{Q}{\rho c_p} \quad (13a)$$

$$\left. \begin{aligned} \frac{\partial t_Q}{\partial y} &= 0 && \text{at } y = a \text{ (insulated wall)} \\ \frac{\partial t_Q}{\partial y} &= 0 && \text{at } y = 0 \text{ (symmetry)} \\ t_Q &= 0 && \text{at } x = 0 \text{ (entrance condition)} \end{aligned} \right\} \quad (13b)$$

and

$$u \frac{\partial t_q}{\partial x} = \frac{\partial}{\partial y} \left[ (\alpha + \epsilon_H) \frac{\partial t_q}{\partial y} \right] \quad (14a)$$

$$\left. \begin{aligned} \frac{\partial t_q}{\partial y} &= \frac{q_w}{\kappa} & \text{at } y = a \text{ (specified heat flux)} \\ \frac{\partial t_q}{\partial y} &= 0 & \text{at } y = 0 \text{ (symmetry)} \\ t_q &= t_i & \text{at } x = 0 \text{ (entrance condition)} \end{aligned} \right\} \quad (14b)$$

Before attempting to solve for the temperatures  $t_Q$  and  $t_q$ , it is convenient to write the preceding equations as follows:

$$U \frac{\partial t_Q}{\partial x'} = \frac{\partial}{\partial y'} \left[ (1 + \lambda_T) \frac{\partial t_Q}{\partial y'} \right] + \frac{Qa^2}{\kappa} \quad (15a)$$

$$\left. \begin{aligned} \frac{\partial t_Q}{\partial y'} &= 0 & \text{at } y' = 0 \text{ and } y' = 1 \\ t_Q &= 0 & \text{at } x' = 0 \end{aligned} \right\} \quad (15b)$$

and

$$U \frac{\partial t_q}{\partial x'} = \frac{\partial}{\partial y'} \left[ (1 + \lambda_T) \frac{\partial t_q}{\partial y'} \right] \quad (16a)$$

$$\left. \begin{aligned} \frac{\partial t_q}{\partial y'} &= \frac{q_w a}{\kappa} & \text{at } y' = 1 \\ \frac{\partial t_q}{\partial y'} &= 0 & \text{at } y' = 0 \\ t_q &= t_i & \text{at } x' = 0 \end{aligned} \right\} \quad (16b)$$

Equation (10), for  $\lambda_T$ , may be inserted into equations (15a) and (16a). It is also convenient to introduce the change of variable

$$r^2 = 1 + (k^2 - 1)(1 - y') \quad (17)$$

which then allows equations (15) and (16) to be written more compactly as

$$U \frac{\partial t_Q}{\partial x'} = \frac{a_o}{r} \frac{\partial}{\partial r} \left( r \frac{\partial t_Q}{\partial r} \right) + \frac{Q a^2}{\kappa} \quad (18a)$$

$$\left. \begin{aligned} \frac{\partial t_Q}{\partial r} &= 0 && \text{at } r = 1 \text{ and } r = k \\ t_Q &= 0 && \text{at } x' = 0 \end{aligned} \right\} \quad (18b)$$

and

$$U \frac{\partial t_q}{\partial x'} = \frac{a_o}{r} \frac{\partial}{\partial r} \left( r \frac{\partial t_q}{\partial r} \right) \quad (19a)$$

$$\left. \begin{aligned} \frac{\partial t_q}{\partial r} &= -\frac{1}{\sqrt{a_o}} \frac{q_w a}{\kappa} && \text{at } r = 1 \\ \frac{\partial t_q}{\partial r} &= 0 && \text{at } r = k \\ t_q &= t_i && \text{at } x' = 0 \end{aligned} \right\} \quad (19b)$$

where  $a_o = (k^2 - 1)^2/4$ . The power-law velocity expression, in terms of the new variable  $r$ , is given by

$$U(r) = \frac{m+1}{(k^2 - 1)^m} (r^2 - 1)^m \quad (20)$$

Equations (18) and (19) complete the formulation of the boundary-value problems. The problem of solving for  $t_Q$  and  $t_q$  is most conveniently attacked by separation of entrance region and fully developed solutions:

$$t_Q = t_{Q,d} + t_{Q,e} \quad (21a)$$

$$t_q = t_{q,d} + t_{q,e} \quad (21b)$$

The two problems as defined by equations (21a) and (21b) are analyzed separately. The solutions are then combined to yield the results for the general case of both internal heat generation and wall heat transfer.

The solution for  $t_Q$  which applies both in the entrance and fully developed regions is

$$\frac{t_Q}{\frac{Qa^2}{\kappa}} = x' + F(r) + \sum_{n=1}^{\infty} A_n R_n(r) e^{-\alpha_n^2 x'} \quad (22)$$

in which the radial function  $F(r)$  is the solution of the mathematical system

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dF}{dr} \right) = \frac{1}{a_o} [U(r) - 1] \quad (23a)$$

$$\frac{dF}{dr} = 0 \quad \text{at } r = 1 \text{ and } r = k \quad (23b)$$

$$\int_1^k U(r) F(r) r \, dr = 0 \quad (23c)$$

The condition at  $r = k$  is automatically satisfied when the condition at  $r = 1$  is satisfied. The last condition is the result of the consideration of an overall energy balance on the fluid in the fully developed region. The  $\alpha_n^2$  and  $R_n$  are, respectively, the eigenvalues and eigenfunctions of the Sturm-Liouville mathematical system

$$\left. \begin{aligned} \frac{1}{r} \frac{d}{dr} \left( r \frac{dR_n}{dr} \right) + \frac{1}{a_o} U(r) \alpha_n^2 R_n &= 0 \\ \frac{dR_n}{dr} &= 0 \quad \text{at } r = 1 \text{ and } r = k \end{aligned} \right\} \quad (24)$$

The coefficients  $A_n$  are determined to satisfy the condition that  $t_Q = 0$  at the channel entrance:

$$A_n = - \frac{\int_1^k U(r) F(r) R_n(r) r \, dr}{\int_1^k U(r) R_n^2(r) r \, dr} \quad (25)$$

Equation (23a) for the radial function  $F(r)$  can be integrated directly. The resulting expression for  $F(r)$  is

$$F(r) = \frac{1}{a_o} \left[ I_1(r) - \frac{(r^2 - 1)}{4} + \frac{1}{2} \ln r \right] + F(1) \quad (26)$$

where

$$F(1) = \frac{1}{a_o \sqrt{a_o}} \left[ \frac{1}{4} I_2(k) - I_3(k) - \frac{1}{2} I_4(k) \right] \quad (27a)$$

$$I_1(r) = \int_1^r \frac{\int_1^\beta U(\alpha) \alpha \, d\alpha}{\beta} d\beta \quad (27b)$$

$$I_2(k) = \int_1^k (r^3 - r) U(r) dr \quad (27c)$$

$$I_3(k) = \int_1^k U(r) I_1(r) r \, dr \quad (27d)$$

$$I_4(k) = \int_1^k U(r) r \ln r \, dr \quad (27e)$$

and  $\alpha$  and  $\beta$  are dummy variables.

Similarly, the solution for  $t_q$  is

$$\frac{t_q - t_i}{\frac{q_w a}{K}} = x' + G(r) + \sum_{n=1}^{\infty} B_n R_n(r) e^{-\alpha_n^2 x'} \quad (28)$$

in which the radial function  $G(r)$  is the solution of the mathematical system

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = \frac{1}{a_o} U(r) \quad (29a)$$

$$\frac{dG}{dr} = -\frac{1}{\sqrt{a_o}} \quad \text{at } r = 1 \quad (29b)$$

$$\frac{dG}{dr} = 0 \quad \text{at } r = k \quad (29c)$$

$$\int_1^k U(r) G(r) r \, dr = 0 \quad (29d)$$

As in the foregoing, the  $\alpha_n^2$  and  $R_n(r)$  are the eigenvalues and eigenfunctions of equation (24), but now the coefficients  $B_n$  are

$$B_n = -\frac{\int_1^k U(r) G(r) R_n(r) r \, dr}{\int_1^k U(r) R_n^2(r) r \, dr} \quad (30)$$

in order to satisfy the condition  $t_q = t_i$  at the channel entrance.

Equation (29a) for the radial function  $G(r)$  can also be integrated directly to yield

$$G(r) = \frac{1}{a_o} \left[ I_1(r) - \sqrt{a_o} \ln r \right] + G(1) \quad (31)$$

where

$$G(1) = -\frac{1}{a_0 \sqrt{a_0}} \left[ I_3(k) - \sqrt{a_0} I_4(k) \right] \quad (32)$$

To complete the solution, it is necessary to find the eigenvalues and eigenfunctions of equation (24) for specified values of the exponent  $m$  in the power-law velocity expression, and to carry out the integrations for the  $A_n$  and  $B_n$  in equations (25) and (30). The determination of these quantities will be considered later.

Combining the results of the previous paragraphs, it is possible to write the solution for the situation where internal heat generation and wall heat transfer occur simultaneously. The temperature distribution for this combined problem, according to equation (12), is found by adding the contributions due to each of the separate problems. The temperature difference  $t(x', r) - t_i$ , which applies both in the entrance and fully developed regions, therefore, can be written as

$$t - t_i = \frac{Qa^2}{K} \left[ x' + F(r) + \sum_{n=1}^{\infty} A_n R_n(r) e^{-\alpha_n^2 x'} \right] + \frac{q_w a}{K} \left[ x' + G(r) + \sum_{n=1}^{\infty} B_n R_n(r) e^{-\alpha_n^2 x'} \right] \quad (33a)$$

It is convenient to rewrite equation (33a) in the nondimensional form

$$\frac{t - t_i}{\frac{Qa^2}{K}} = (1 + R)x' + F(r) + \sum_{n=1}^{\infty} A_n R_n(r) e^{-\alpha_n^2 x'} + R \left[ G(r) + \sum_{n=1}^{\infty} B_n R_n(r) e^{-\alpha_n^2 x'} \right] \quad (33b)$$

where

$$R = \frac{q_w}{Qa} \quad (34)$$

The heat-flux parameter  $R$  is essentially the ratio of the heat transferred at the channel walls to the heat generated internally.

Of particular practical interest is the wall temperature variation corresponding to internal heat generation and wall heat transfer. This quantity can be found from equation (33b) by evaluating the equation at  $r = 1$ :

$$\frac{t_w - t_i}{\frac{Qa^2}{K}} = (1 + R)x' + F(1) + \sum_{n=1}^{\infty} \bar{A}_n e^{-\alpha_n^2 x'} + R \left[ G(1) + \sum_{n=1}^{\infty} \bar{B}_n e^{-\alpha_n^2 x'} \right] \quad (35)$$

Another result of practical engineering importance is the variation in the wall- to bulk- temperature difference. The bulk temperature  $t_b$  is given by

$$t_b(x') = t_i + \left( \frac{Qa^2}{K} + \frac{q_w a}{K} \right) x' \quad (36a)$$

or, alternately,

$$\frac{t_b - t_i}{\frac{Qa^2}{K}} = (1 + R)x' \quad (36b)$$

The bulk temperature rises in a linear fashion along the channel. The difference between the wall and bulk temperatures at all stations along the channel is given by

$$\frac{t_w - t_b}{\frac{Qa^2}{K}} = F(1) + \sum_{n=1}^{\infty} \bar{A}_n e^{-\alpha_n^2 x'} + R \left[ G(1) + \sum_{n=1}^{\infty} \bar{B}_n e^{-\alpha_n^2 x'} \right] \quad (37)$$

A useful form of the wall temperature results, given by equation (37), is obtained by separate consideration of the insulated or adiabatic wall ( $R = 0$ ) and nonadiabatic wall ( $R \neq 0$ ) cases.

The wall- to bulk-temperature difference for the insulated wall case, denoted by  $(t_w - t_b)_0$ , is found by evaluating equation (37) with  $R$  set equal to zero:

$$\frac{(t_w - t_b)_0}{\frac{Qa^2}{K}} = F(1) + \sum_{n=1}^{\infty} \bar{A}_n e^{-\alpha_n^2 x'} \quad (38)$$



By combining equations (37) and (38), the following expression for the ratio  $(t_w - t_b)/(t_w - t_b)_0$  is obtained:

$$\frac{t_w - t_b}{(t_w - t_b)_0} = 1 + R \frac{G(1) + \sum_{n=1}^{\infty} \bar{B}_n e^{-\alpha_n^2 x'}}{F(1) + \sum_{n=1}^{\infty} \bar{A}_n e^{-\alpha_n^2 x'}} \quad (39)$$

The ratio given by equation (39) depends separately on the velocity distribution and the diffusivity parameter, as well as on  $x'$ . The departure of  $(t_w - t_b)/(t_w - t_b)_0$  from unity is a measure of wall heat-transfer effects.

The wall- to bulk-temperature difference for the fully developed situation ( $x' \rightarrow \infty$ ) is found from equation (37) to be

$$\frac{(t_w - t_b)_d}{\frac{Qa^2}{\kappa}} = F(1) + RG(1) \quad (40)$$

A convenient rephrasing of equation (37) may then be carried out by introducing the fully developed wall- to bulk-temperature difference, which yields

$$\frac{t_w - t_b}{(t_w - t_b)_d} = 1 + \frac{F(1) + \sum_{n=1}^{\infty} \bar{A}_n e^{-\alpha_n^2 x'}}{F(1) + RG(1)} + R \frac{G(1) + \sum_{n=1}^{\infty} \bar{B}_n e^{-\alpha_n^2 x'}}{F(1) + RG(1)} \quad (41)$$

The departure of  $(t_w - t_b)/(t_w - t_b)_d$  from unity is a measure of the thermal entrance effects.

The heat-transfer characteristics of both laminar and turbulent flow are customarily represented in terms of a heat-transfer coefficient  $h \equiv q_w/(t_w - t_b)$  and a Nusselt number  $Nu \equiv hD_H/\kappa$ , where  $D_H$  is the hydraulic diameter ( $D_H = 4a$  for the parallel-plate channel). With these definitions, it follows from equation (37) that

$$\text{Nu} = \frac{4}{S \left[ F(1) + \sum_{n=1}^{\infty} \bar{A}_n e^{-\alpha_n^2 x'} \right] + G(1) + \sum_{n=1}^{\infty} \bar{B}_n e^{-\alpha_n^2 x'}} \quad (42)$$

where

$$S = \frac{Qa}{q_w} = \frac{1}{R} \quad (43)$$

The Nusselt number depends on the important parameters  $k$  and  $m$ , the longitudinal position  $x'$ , and the inverse heat-flux ratio  $S$ . The Nusselt number in the absence of internal heat generation, denoted by  $\text{Nu}_0$ , is obtained by evaluating equation (42) with  $S = 0$ . The result is given by

$$\text{Nu}_0 = \frac{4}{G(1) + \sum_{n=1}^{\infty} \bar{B}_n e^{-\alpha_n^2 x'}} \quad (44)$$

As a matter of general interest, the Nusselt number  $\text{Nu}_0$  in the fully developed region, denoted by  $\text{Nu}_{0,d}$ , is considered. This is obtained from equation (44) by considering the limit  $x' \rightarrow \infty$ :

$$\text{Nu}_{0,d} = \frac{4}{G(1)} \quad (45)$$

Use will be made of the fully developed Nusselt number in the section entitled RESULTS AND DISCUSSION.

### Limiting Case: Diffusivity Parameter $k = 1$

The analysis of the preceding section led to the determination of the wall temperature distribution and heat-transfer characteristics with transverse heat diffusion occurring by turbulent eddying as well as by molecular conduction. It is of interest to examine the limiting case where the thermal eddy diffusivity is negligible compared with the thermal

molecular diffusivity. In this situation, transverse heat transfer is by molecular conduction only; this corresponds to the limiting value for  $k$  of one. A number of equations presented in the preceding section assume indeterminate forms for the limiting case  $k = 1$ , and the evaluation of these results is a problem of considerable mathematical complexity. It is more convenient in this connection to reconsider equations (15) and (16) and to set  $\lambda_T = 0$  therein. The solution of the subsequent boundary value problems then yields the temperature distribution and heat-transfer characteristics in the absence of heat conduction by turbulent eddying. The methods of solution are similar to those outlined in the earlier portion of this investigation; therefore the details of the derivation are omitted. The solutions for the wall- to bulk-temperature difference  $t_w - t_b$  and the bulk temperature change  $t_b - t_i$  are

$$\frac{t_w - t_b}{\frac{Qa^2}{K}} = \tilde{F}(1) + \sum_{n=1}^{\infty} \bar{C}_n e^{-\beta_n^2 x'} + R \left[ \tilde{G}(1) + \sum_{n=1}^{\infty} \bar{D}_n e^{-\beta_n^2 x'} \right] \quad (46)$$

$$\frac{t_b - t_i}{\frac{Qa^2}{K}} = (1 + R)x' \quad (47)$$

in which

$$\tilde{F}(1) = \tilde{I}_1(1) - \frac{1}{2} [1 - \tilde{I}_2(1)] - \tilde{I}_3(1) \quad (48)$$

$$\tilde{G}(1) = \tilde{I}_1(1) - \tilde{I}_3(1) \quad (49)$$

$$\tilde{I}_1(1) = \int_0^1 \left[ \int_0^\xi U(\alpha) d\alpha \right] d\xi \quad (50a)$$

$$\tilde{I}_2(1) = \int_0^1 U(\xi) \xi^2 d\xi \quad (50b)$$

$$\tilde{I}_3(1) = \int_0^1 U(\tau) \tilde{I}_1(\tau) d\tau \quad (50c)$$

and

$$\tilde{I}_1(\tau) = \int_0^\tau \left[ \int_0^\xi U(\alpha) d\alpha \right] d\xi \quad (50d)$$

The variables  $\alpha$ ,  $\xi$ , and  $\tau$  are dummy variables. The coefficients  $\bar{C}_n \equiv C_n Y_n(1)$  and  $\bar{D}_n \equiv D_n Y_n(1)$  are given by

$$\bar{C}_n = - \frac{Y_n(1) \int_0^1 Y_n(y') dy'}{\beta_n^2 \int_0^1 U(y') Y_n^2(y') dy'} \quad (51)$$

$$\bar{D}_n = - \frac{Y_n^2(1)}{\beta_n^2 \int_0^1 U(y') Y_n^2(y') dy'} \quad (52)$$

The  $\beta_n^2$  and  $Y_n$  appearing in equations (46), (51), and (52) are, respectively, the eigenvalues and eigenfunctions of the mathematical system

$$\left. \begin{aligned} \frac{d^2 Y_n}{dy'^2} + U(y') \beta_n^2 Y_n &= 0 \\ \frac{dY_n}{dy'} &= 0 \text{ at } y' = 0 \text{ and } y' = 1 \end{aligned} \right\} \quad (53)$$

The important wall temperature ratios  $(t_w - t_b)/(t_w - t_b)_0$  and  $(t_w - t_b)/(t_w - t_b)_d$  are readily obtained from equation (46) as

$$\frac{t_w - t_b}{(t_w - t_b)_0} = 1 + R \frac{\tilde{G}(1) + \sum_{n=1}^{\infty} \bar{D}_n e^{-\beta_n^2 x'}}{\tilde{F}(1) + \sum_{n=1}^{\infty} \bar{C}_n e^{-\beta_n^2 x'}} \quad (54)$$

$$\frac{t_w - t_b}{(t_w - t_b)_d} = 1 + \frac{\tilde{F}(1) + \sum_{n=1}^{\infty} \bar{C}_n e^{-\beta_n^2 x'} + R \left[ \tilde{G}(1) + \sum_{n=1}^{\infty} \bar{D}_n e^{-\beta_n^2 x'} \right]}{\tilde{F}(1) + R\tilde{G}(1)} \quad (55)$$

The Nusselt number in the present situation is given by the following equation:

$$Nu = \frac{4}{S \left[ \tilde{F}(1) + \sum_{n=1}^{\infty} \bar{C}_n e^{-\beta_n^2 x'} \right] + \tilde{G}(1) + \sum_{n=1}^{\infty} \bar{D}_n e^{-\beta_n^2 x'}} \quad (56)$$

Then the Nusselt number in the absence of internal heat generation is obtained from

$$Nu_0 = \frac{4}{\tilde{G}(1) + \sum_{n=1}^{\infty} \bar{D}_n e^{-\beta_n^2 x'}} \quad (57)$$

from which it follows that

$$Nu_{0,d} = \frac{4}{\tilde{G}(1)} \quad (58)$$

## Transverse Distribution Functions $R(r)$ , $Y(y')$

Attention is now directed to the Sturm-Liouville eigenvalue problems (eqs. (24) and (53)). Consideration will be given first to the determination of the function  $R(r)$  which is the solution to equation (24). To obtain a solution of equation (24), it is necessary that the variation of  $U$  with  $r$  be specified. When the velocity profile (eq. (20)) is introduced into equation (24), the governing equation and boundary conditions for  $R(r)$  are

$$\left. \begin{aligned} \frac{1}{r} \frac{d}{dr} \left( r \frac{dR_n}{dr} \right) + \lambda_n^2 (r^2 - 1)^m R_n &= 0 \\ \frac{dR_n}{dr} &= 0 \text{ at } r = 1 \text{ and } r = k \end{aligned} \right\} \quad (59)$$

where

$$\lambda_n^2 = \frac{4(m+1)\alpha_n^2}{(k^2 - 1)^{m+2}}$$

A different set of eigenvalues and eigenfunctions will be obtained for each independent value of  $k$  and  $m$ . For  $m = 0$  there results a Bessel equation which, in general, has solutions in terms of ordinary Bessel functions of integral order. For this case, the first five eigenvalues  $\alpha_n^2$  and coefficients  $\bar{B}_n$  have been given in reference 2. (For  $m = 0$ , the coefficients  $\bar{A}_n$  are equal to 0). For positive, fractional values of  $m$ , an analytical solution for equation (59) in terms of tabulated functions for arbitrary values of  $k$  does not appear possible. Therefore, it was decided to integrate equation (59) numerically for a value of  $m$  of  $1/7$  by the Runge-Kutta method and to determine the desired eigenvalues by trial and error. In addition, the coefficients  $\bar{A}_n$  and  $\bar{B}_n$ , given by

$$\bar{A}_n = - \frac{4R_n(0) \int_0^{(k^2-1)} R_n(s) ds}{(k^2 - 1)^2 \lambda_n^2 \int_0^{(k^2-1)} s^m R_n^2(s) ds} \quad (60)$$

$$\bar{B}_n = - \frac{4R_n^2(0)}{(k^2 - 1)\lambda_n^2 \int_0^{(k^2-1)} s^m R_n^2(s) ds} \quad (61)$$

where the change of variable  $r^2 - 1 = s$  has been introduced, are to be obtained by numerical integration. Solutions were carried out for values of the diffusivity parameter  $k$  of 1.25, 1.67, 2.50, and 5.00. The first four eigenvalues  $\alpha_n^2$  and coefficients  $\bar{A}_n$  and  $\bar{B}_n$  thus obtained are listed in tables I and II, respectively.

TABLE I. - LISTING OF EIGENVALUES

$\alpha_n^2$  FOR 1/7-POWER-LAW

VELOCITY EXPRESSION

Diffusivity parameter, k	Eigenvalue			
	$\alpha_1^2$	$\alpha_2^2$	$\alpha_3^2$	$\alpha_4^2$
1.25	13.06	51.25	114.2	203.1
1.67	18.89	73.25	163.0	288.9
2.50	34.95	131.5	290.9	514.5
5.00	115.8	412.0	894.9	1566

TABLE II. - LISTING OF COEFFICIENTS FOR 1/7-POWER-LAW VELOCITY EXPRESSION

Diffusivity parameter, k	Coefficient							
	$-\bar{A}_1$	$-\bar{A}_2$	$-\bar{A}_3$	$-\bar{A}_4$	$-\bar{B}_1$	$-\bar{B}_2$	$-\bar{B}_3$	$-\bar{B}_4$
1.25	0.01379	0.001478	0.0005431	0.0002160	0.18784	0.05045	0.02322	0.01336
1.67	.01052	.001071	.0004058	.0001571	.15035	.04187	.01945	.01124
2.50	.00659	.000646	.0002561	.0000956	.10072	.03053	.01427	.00848
5.00	.00237	.000238	.0001021	.0000365	.04009	.01484	.00764	.00463

There remains to consider in this section the eigenvalues and eigenfunctions of equation (53) and to carry out the evaluation for the coefficients  $\bar{C}_n$  and  $\bar{D}_n$  in equations (51) and (52). When the velocity profile (eq. (4)) is introduced into equation (53) and when the change in variable  $\eta = 1 - y'$  is introduced, the governing equation and boundary conditions for  $Y(\eta)$  are found to be

$$\left. \begin{aligned} \frac{d^2 Y_n}{d\eta^2} + \eta^m \omega_n^2 Y_n &= 0 \\ \frac{dY_n}{d\eta} &= 0 \text{ at } \eta = 0 \text{ and } \eta = 1 \end{aligned} \right\} \quad (62)$$

where  $\omega_n^2 = (m+1)\beta_n^2$ . A different set of eigenvalues and eigenfunctions will be obtained for each value of  $m$ . For the limiting case  $m = 0$ , a solution of equation (62) that satisfies the boundary conditions is found readily to be

$$\begin{aligned} Y_n(\eta) &= E_1 \cos[n\pi(1 - \eta)] \\ &= E_1 \cos n\pi\eta \end{aligned} \quad (63a)$$

where the eigenvalues  $\omega_n^2 = \beta_n^2$  are given by

$$\omega_n^2 = (n\pi)^2 \quad (63b)$$

The result is in agreement with that in reference 1. The constant  $E_1$  is an arbitrary constant. It is readily shown that, in addition, the coefficients  $\bar{D}_n$  are given as

$$\bar{D}_n = -\frac{2}{(n\pi)^2} \quad (63c)$$

The solution to equation (62) for positive fractional values of  $m$  remains to be determined. It can be shown that the general solution to the differential equation (62) is

$$Y(\eta) = \eta^{1/2} \left[ E_2 J_{1/(m+2)} \left( \frac{2}{m+2} \omega \eta^{(m+2)/2} \right) + E_3 Y_{1/(m+2)} \left( \frac{2}{m+2} \omega \eta^{(m+2)/2} \right) \right] \quad (64)$$

where  $J_{1/(m+2)}$  and  $Y_{1/(m+2)}$  are the  $1/(m+2)$  order Bessel functions of the first and second kind, respectively, and  $E_2$  and  $E_3$  are arbitrary constants. The solution of equation (64) involves Bessel functions of fractional order. The eigenvalues  $\omega_n^2$  are obtained by requiring the general solution to satisfy the boundary conditions. If characteristic numbers and zeros were available for these fractional-order functions, then an analytical solution for the important eigenvalues and coefficients would be possible. In



particular, a value of the exponent  $m$  of  $1/7$  is to be used in the present evaluation of the solution. However, the author was unable to find any tabulation of Bessel functions (and derivatives) of the fractional order  $7/15$ ; an analytical solution for  $Y(\eta)$  was, therefore, not possible and a numerical technique (Runge-Kutta method) had to be employed.

The first four eigenvalues  $\beta_n^2 = (7/8)\omega_n^2$  and coefficients  $\bar{C}_n$  and  $\bar{D}_n$ , alternately given as

$$\bar{C}_n = - \frac{Y_n(0) \int_0^1 Y_n(\eta) d\eta}{\omega_n^2 \int_0^1 \eta^{1/7} Y_n^2(\eta) d\eta} \quad (65)$$

$$\bar{D}_n = - \frac{Y_n^2(0)}{\omega_n^2 \int_0^1 \eta^{1/7} Y_n^2(\eta) d\eta} \quad (66)$$

were evaluated by utilizing numerical integration. The eigenvalues and coefficients are listed in table III.

TABLE III. - LISTING OF EIGENVALUES  
AND COEFFICIENTS FOR  $1/7$ -POWER-  
LAW VELOCITY EXPRESSION

Index, n	Eigenvalue, $\beta_n^2$	Coefficient, $-\bar{C}_n$	Coefficient, $-\bar{D}_n$
1	10.22	0.0163500	0.21510
2	40.31	.0018452	.05702
3	90.23	.0006639	.02615
4	160.0	.0002694	.01503

Constants  $F(1)$ ,  $G(1)$ ,  $\tilde{F}(1)$ , and  $\tilde{G}(1)$

To complete the solution for the wall temperature distribution and heat-transfer characteristics it is necessary to carry out the integrations for  $F(1)$  and  $G(1)$  in equations (27a) and (32) and for  $\tilde{F}(1)$  and  $\tilde{G}(1)$  in equations (48) and (49). Some of the integration is elementary; the remainder requires changes of variable or repeated integration by

parts. The considerable amount of analytical detail is omitted, with the results presented as follows:

$$F(1) = \frac{m+1}{(m+2)(k^2-1)} - \frac{4(m+1)}{(k^2-1)^{2m+3}} \int_1^k (r^2-1)^m \left[ \int_1^r \frac{(\xi^2-1)^{m+1}}{\xi} d\xi \right] r dr$$

$$- \frac{2}{(k^2-1)^2} \ln k + \frac{1}{(k^2-1)^{m+3}} \int_1^{k^2} \frac{(\varphi-1)^{m+1}}{\varphi} d\varphi \quad (67a)$$

$$G(1) = \frac{2}{(k^2-1)} \ln k - \frac{4(m+1)}{(k^2-1)^{2m+3}} \int_1^k (r^2-1)^m \left[ \int_1^r \frac{(\xi^2-1)^{m+1}}{\xi} d\xi \right] r dr$$

$$- \frac{1}{(k^2-1)^{m+2}} \int_1^{k^2} \frac{(\varphi-1)^{m+1}}{\varphi} d\varphi \quad (67b)$$

$$\tilde{F}(1) = \frac{m+1}{2(m+3)} - \frac{m+1}{(m+2)(2m+3)} \quad (67c)$$

$$\tilde{G}(1) = \frac{2(m+1)^2}{(m+2)(2m+3)} \quad (67d)$$

For the special case  $m = 0$  (uniform velocity profile), it is easy to verify that these constants reduce to

$$F(1) = \tilde{F}(1) = 0 \quad (68a)$$

$$G(1) = \frac{2k^2 \ln k - (k^2-1)^2 - \frac{1}{2}(k^4-1)}{(k^2-1)^3} \quad (68b)$$

$$\tilde{G}(1) = \frac{1}{3}$$

These results have been obtained earlier in reference 2. For positive fractional values of  $m$ , an analytical solution for  $F(1)$  and  $G(1)$  is not possible because the integrals contained therein are intractable, and numerical integration has to be employed. Solutions were carried out on an electronic computer for a value of  $m$  of  $1/7$  and for parametric values of  $k$  of 1.25, 1.67, 2.50, and 5.00. Numerical values of  $F(1)$  and  $G(1)$ , in addition to  $\tilde{F}(1)$  and  $\tilde{G}(1)$  evaluated from equations (67c) and (67d) for  $m = 1/7$ , are listed in table IV.

TABLE IV. - LISTING OF CONSTANTS  
FOR  $1/7$ -POWER-LAW VELOCITY  
EXPRESSION

Diffusivity parameter, k	Constant	
	$\tilde{F}(1)$	$\tilde{G}(1)$
1.00	0.01950	0.3711
	F(1)	G(1)
1.25	0.01632	0.3263
1.67	.01236	.2665
2.50	.00705	.1877
5.00	.00279	.0867

## RESULTS AND DISCUSSION

With the numerical information in tables I to IV, the longitudinal variations of the dimensionless adiabatic or insulated wall temperatures were evaluated from the analytical solution (eq. (37)) and the results are plotted in figure 3. In interpreting this figure, it is important to note that the Reynolds and Prandtl numbers appear both in the parameter  $k$  and in the abscissa. The information given in this plot permits evaluation of the insulated wall- to bulk-temperature difference at various stations along the channel as a function of Reynolds and Prandtl number. These results apply for a value of the power-law velocity exponent of  $1/7$ .

Another quantity which is of practical interest is the fluid bulk temperature variation along the length of the channel. The bulk temperature is given relative to the temperature of the fluid at the entrance to the channel. The dimensionless bulk temperature

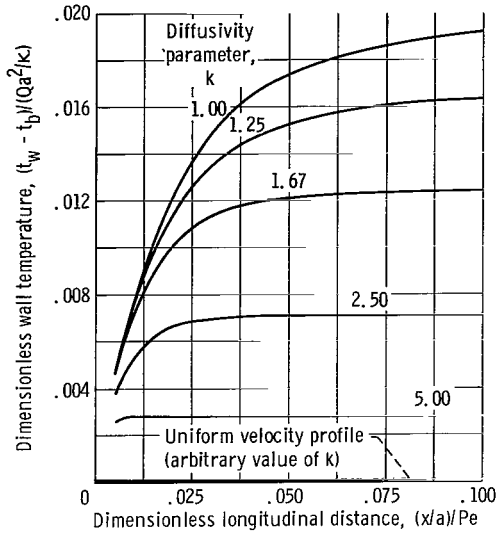


Figure 3. - Wall temperature results for internal heat generation in channel with insulated walls. Power-law velocity exponent,  $1/7$ .

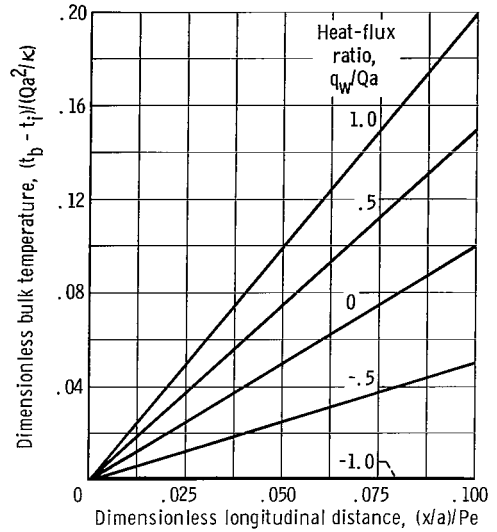


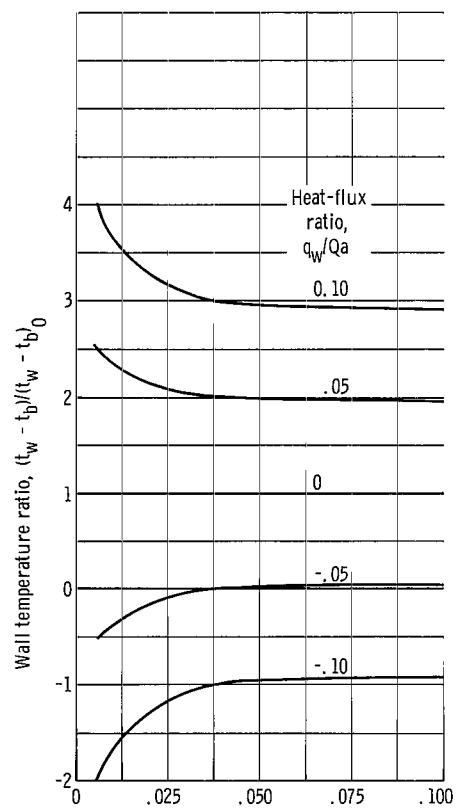
Figure 4. - Bulk temperature results for internal heat generation with wall heat transfer.

given by equation (36b) has been plotted in figure 4 for parametric values of the heat-flux parameter  $R$ .

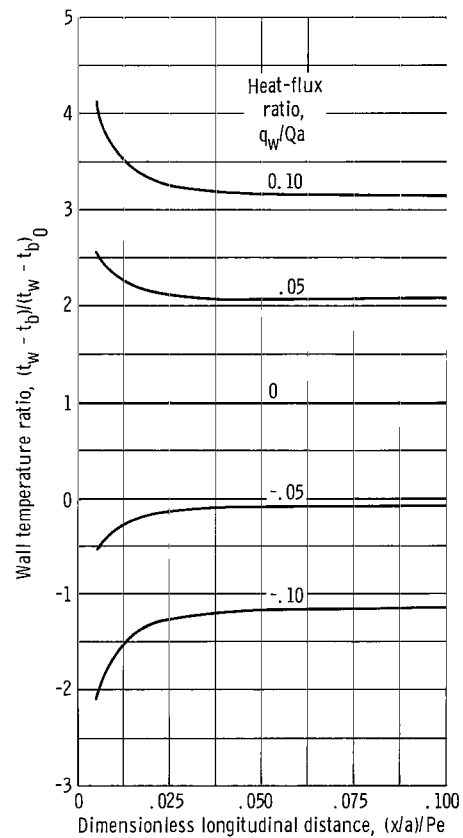
Positive and negative values of the parameter  $R$  are considered in the figure. In the present analysis,  $Q$  is taken to be positive (a heat source). A positive value of  $R$ , therefore, implies that  $q_w$  is positive, or that heat is being transferred from the walls to the fluid. A negative value of  $R$ , on the other hand, implies that  $q_w$  is negative and, therefore, that heat is being transferred from the fluid to the walls. The case of  $R = 0$  corresponds to the thermally insulated wall. For the special value of  $R = -1.0$ , the fluid flows with a bulk temperature which is unchanging with length. For any other arbitrary value of  $R$ , the bulk temperature rises (or falls) in a linear fashion along the channel.

The ratio of nonadiabatic to adiabatic wall- to bulk-temperature differences (eq. (39)) is plotted in figure 5 as a function of the dimensionless axial distance along the channel for parametric values of  $R$  using the numerical data listed in tables I to IV. The information given on these plots, used in conjunction with figure 3, permits evaluation of the wall- to bulk-temperature difference at various stations along the heated channel walls. It is noted that, for some negative values of the heat flux ratio  $R$ , the wall- to bulk-temperature difference may be negative, which means that  $t_b$  is larger than  $t_w$ . This is understandable, however, if it is recalled that  $t_w$  is a local value along the wall, while  $t_b$  is an average value over the entire cross section.

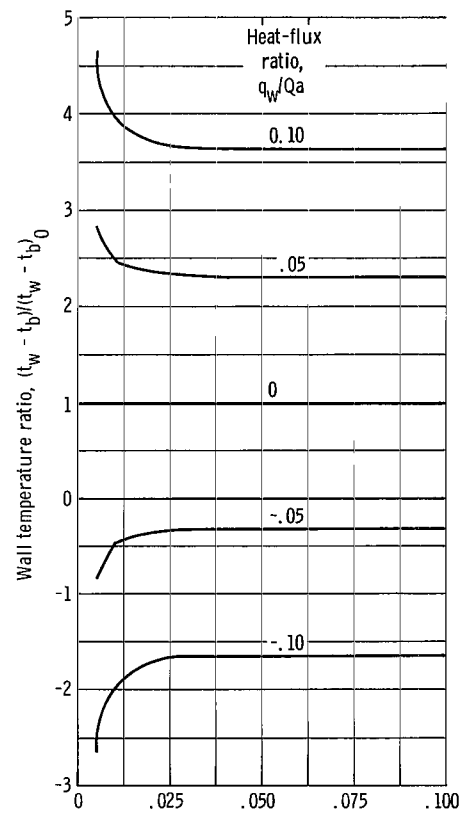
The wall- to bulk-temperature difference divided by the fully developed difference is plotted in figure 6 for flow in an insulated channel and parametric values of  $k$ . The ratio is given by equations (41) and (55) with  $R$  set equal to zero therein. The information given on this plot, used in conjunction with figure 3, permits evaluation of the wall- to



(a) Diffusivity parameter, 1.00.



(b) Diffusivity parameter, 1.67.



(c) Diffusivity parameter, 2.50.

Figure 5. - Wall temperature ratios for internal heat generation with wall heat transfer. Power-law velocity exponent,  $1/7$ .

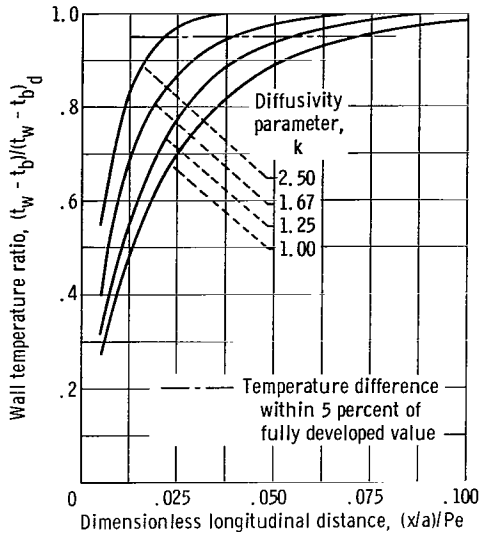


Figure 6. - Wall temperature ratios for internal heat generation in channel with insulated walls. Power-law velocity exponent,  $1/7$ .

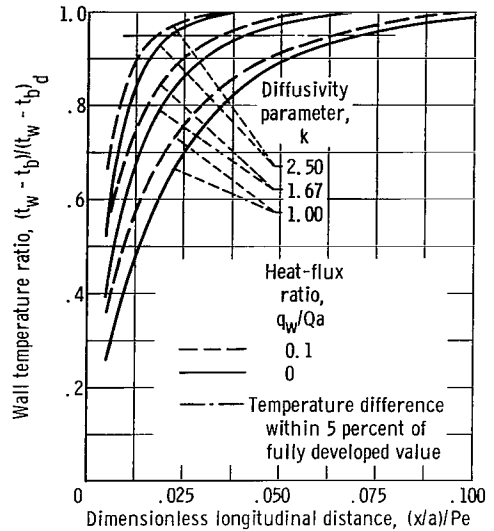


Figure 7. - Wall temperature ratios for internal heat generation in channel with wall heat transfer. Power-law velocity exponent,  $1/7$ .

bulk-temperature difference at locations along the thermally insulated walls. Thermal entrance length is commonly defined as the heated length required for  $t_w - t_b$  to approach to within 5 percent of the fully developed value. A dashed line is drawn in figure 6 to facilitate finding the entrance length. For example, for  $Re = 15\,000$  and  $Pr = 0.01$  ( $Pe = 150$  and  $k = 1.25$  for  $\bar{\psi} = 1$ ), the dimensionless entrance length  $x/a$  is approximately 8, while for  $Re = 175\,000$  and the same Prandtl number,  $Pr = 0.01$  ( $Pe = 1750$  and  $k = 2.50$  for  $\bar{\psi} = 1$ ), the entrance length  $x/a$  is approximately 37. Therefore the entrance length increases as the Reynolds number increases.

The thermal entrance length is also influenced by heat addition at the channel walls for various values of  $k$ . This is shown in figure 7, which gives the wall temperature ratio in the thermal entrance region as a function of parametric values of  $k$  and  $R$ . Then, for example, for  $Re = 175\,000$  and  $Pr = 0.01$ , the entrance length, for  $R = 0.1$ , is reduced to approximately 32. An approximately 13-percent reduction occurs in the entrance length with the given conditions when  $R$  varies from 0 to 0.1.

In order to assess the accuracy of the present results, it is desirable to compare them with existing numerical and experimental data. To my knowledge, no heat-transfer measurements for turbulent liquid-metal flows between parallel plates with heat sources in the fluid stream are available for comparison. Reference 18 presents an analytical study of turbulent heat transfer in fully developed flow between parallel plates with internal heat sources. The fluid had a range of Prandtl number from 1 to 100. These results, however, are not applicable to liquid metals whose high thermal conductivities give a range of Prandtl numbers from approximately 0.001 to 0.1.

As mentioned previously, it is customary to represent heat-transfer results in terms of a Nusselt number. Nusselt numbers in the absence of internal heat generation  $Nu_0$  (given by eq. (44) for  $k > 1$  and by eq. (57) for  $k = 1$ ) have been evaluated numerically for a power-law velocity exponent  $m$  of  $1/7$  and are plotted as solid lines in figure 8 for parametric values of  $k$ . Also shown in the figure as dashed lines are Nusselt-number results for a power-law velocity exponent of zero (slug flow). It is noted that the Nusselt-number results for the one-seventh power-law velocity expression are smaller than those for the uniform velocity expression. This finding is in accordance with the results for parallel plates with uniform wall temperatures and no internal heat sources (ref. 9).

It is of practical interest to examine the Nusselt numbers in the fully developed region,  $Nu_{0,d}$ . These results, given by equation (45) for  $k > 1$  and in equation (58) for  $k = 1$  are illustrated in figure 9, where the fully developed Nusselt number is plotted as a function of the parameter  $k$ . Results for the uniform velocity profile and analytical data from references 19 and 20 also appear in figure 9.

The numerical data of references 19 and 20 are discussed in some detail in reference 2. The data in reference 19 is based on experiments in the range  $1.25 \times 10^4 \leq Re \leq 1.35 \times 10^6$ ,  $Pr = 0.0074$ , while that in reference 20 is given for the range  $2000 \leq Re \leq 10^7$ ,  $0 \leq Pr \leq 1.0$ . The Reynolds and Prandtl numbers to which the Nusselt numbers in these investigations correspond were converted to the equivalent values of the parameter  $k$  (calculated for  $\bar{\psi} = \psi = 1$ ). It is evident that the fully developed Nusselt numbers based on the simplified velocity and eddy diffusivity profiles are in good agreement with other numerical analyses.

In view of the foregoing comparison, it may be concluded that the wall-temperature

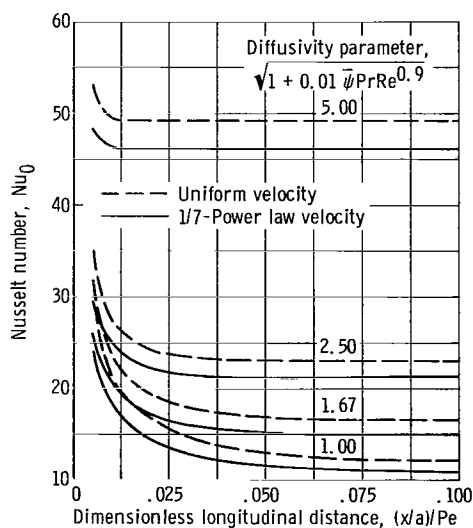


Figure 8. - Nusselt number results for uniform wall heat flux without internal heat sources.

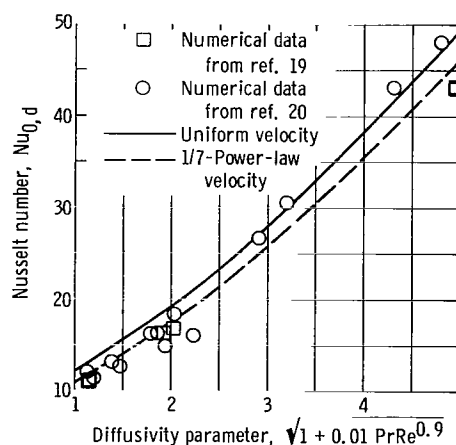


Figure 9 - Comparison of analytical solution and numerical data for fully developed Nusselt numbers with wall heat transfer and no internal heat generation. Average eddy diffusivity ratio, 1.

results based on these profiles have an accuracy adequate for design purposes and can be used to estimate the combined effects of internal heat generation and wall heat transfer.

It is of interest to consider the effect of internal heat generation on the Nusselt number. Consideration will be given here to the fully developed heat-transfer situation, thereby eliminating  $x$  as an additional variable. If the velocity profile is assumed to be uniform, as, for example, at high Reynolds number, then the Nusselt number is independent of the strength of the internal heat source, and the result  $Nu = Nu_0$  (with  $Nu_0$  given by eqs. (44) and (57)) is valid for all values of the inverse heat-flux ratio  $S$ .

If, on the other hand, the velocity profile is assumed given by a  $1/7$  power law, then the effect on the Nusselt number of internal heat generation is determined from equations (42) and (56). Using the numerical data in tables I to IV, the Nusselt numbers  $Nu/Nu_0$  have been plotted in figure 10 as a function of positive values of the inverse heat-flux ratio  $S$  for parametric values of the diffusivity parameter  $k$ . It is evident that the effect of internal heat generation is always to decrease the Nusselt number below its value in the absence of internal heat generation for all values of  $k$ . For given heat-source strength, or inverse heat-flux ratio  $S$ , the reduction in Nusselt number is greatest when transverse heat transfer occurs solely by molecular conduction. The presence of turbulent eddying in the liquid metal causes a smaller reduction in Nusselt number for a given positive value of  $S$ .

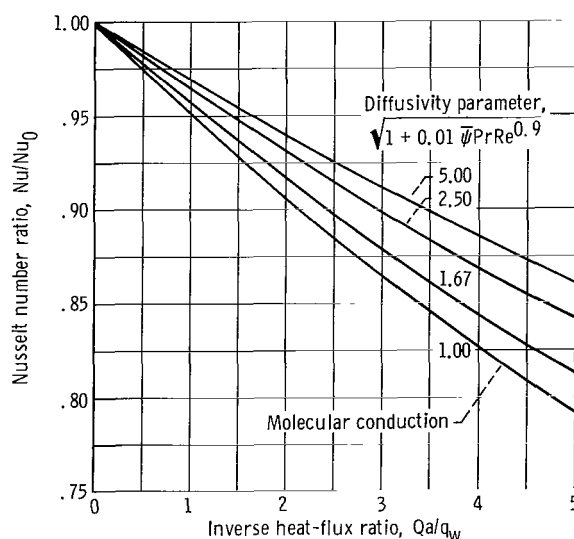


Figure 10. - Variation of fully developed Nusselt number with internal heat source strength for  $1/7$ -power-law velocity distribution.



## CONCLUSIONS

Solutions have been obtained for heat transfer to a liquid metal with an internal heat source and flowing turbulently between parallel plates. The power-law velocity model for the turbulent velocity distribution was utilized for the analyses; a value of the exponent of  $1/7$  was used in the evaluation of the solutions. The effect of transverse eddy diffusivity variations has been included through the use of an idealized eddy diffusivity function. The wall temperatures and Nusselt numbers in both the entrance and fully developed regions can be obtained including the effects of internal heat generation in the fluid and heat transfer at the channel walls.

Some of the characteristics of turbulent liquid-metal heat transfer with internal heat sources and wall heat transfer and with negligible axial conduction can be summarized as follows:

1. The wall- to bulk-temperature difference is affected by the presence of internal heat generation when the velocity profile is approximated by a  $1/7$ -power law. The temperature difference is unaffected by the presence of internal heat generation, however, when the turbulent velocity profile is approximated by a uniform distribution over the channel cross-section.
2. The thermal entrance length for uniform internal heat generation in an insulated parallel-plate channel and for a given Prandtl number is increased as the Reynolds number is increased. For a given Reynolds number, Prandtl number, and heat-source strength, the effect of heat addition at the channel walls is to diminish the entrance length.
3. The effect of internal heat generation is to decrease the Nusselt number below its value in the absence of internal heat generation when the velocity profile is approximated by a  $1/7$ -power law. For a given heat source strength and wall heat flux to the fluid, the Nusselt number reduction is greatest when transverse heat transfer is entirely by molecular conduction.

A comparison of a few of these results with those of others has shown good agreement and, hence, supports the physical model employed. The results should be adequate and useful for preliminary design purposes.

Lewis Research Center,  
National Aeronautics and Space Administration,  
Cleveland, Ohio, September 28, 1967,  
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## APPENDIX - SYMBOLS

$A$	hydrodynamic parameter
$A_n$	coefficients in series expansion of temperature for case of internal heat generation in insulated channel
$\bar{A}_n$	coefficients defined by product $A_n R_n(1)$
$a$	half-height of channel
$a_0$	constant, $(k^2 - 1)^2/4$
$B_n$	coefficients in series expansion of temperature for case of wall heat transfer and no internal heat generation
$\bar{B}_n$	coefficients defined by product $B_n R_n(1)$
$C_n$	coefficients in series expansion of temperature for case of internal heat generation in insulated channel
$\bar{C}_n$	coefficients defined by product $C_n Y_n(1)$
$c_p$	specific heat at constant pressure
$D_H$	hydraulic diameter of channel, $4a$
$D_n$	coefficients in series expansion of temperature for case of wall heat transfer and no internal heat generation
$\bar{D}_n$	coefficients defined by product $D_n Y_n(1)$
$E_1, E_2, E_3$	arbitrary constants
$F(r)$	transverse temperature distribution in fully developed region for case of internal heat generation in insulated channel
$\tilde{F}(y')$	transverse temperature distribution in fully developed region for case of internal heat generation in insulated channel for limiting value of $k = 1$
$G(r)$	transverse temperature distribution in fully developed region for case of wall heat transfer and no internal heat generation
$\tilde{G}(y')$	transverse temperature distribution in fully developed region for case of wall heat transfer and no internal heat generation for limiting value of $k = 1$
$h$	heat transfer coefficient, $q_w/(t_w - t_b)$
$I_1(r)$	integral defined by eq. (27b)
$I_2(k)$	integral defined by eq. (27c)

$I_3(k)$	integral defined by eq. (27d)
$I_4(k)$	integral defined by eq. (27e)
$\tilde{I}_1(\tau)$	integral defined by eq. (50d)
$\tilde{I}_2(1)$	integral defined by eq. (50b)
$\tilde{I}_3(1)$	integral defined by eq. (50c)
$J_{1/(m+2)}$	Bessel function of the first kind and order $1/(m + 2)$
$k$	diffusivity parameter, $\sqrt{1 + 0.01 \bar{\psi} \text{PrRe}^{0.9}}$
$m$	exponent in power-velocity expression
$Nu$	Nusselt number, $hD_H/\kappa$
$Pé$	Péclet number, $\text{RePr}$
$\text{Pr}$	Prandtl number, $\nu/\alpha$
$Q$	heat generation rate/volume
$q_w$	prescribed wall heat flux
$R$	heat-flux ratio, $q_w/Qa$
$\text{Re}$	Reynolds number, $4\bar{u}a/\nu$
$R(r)$	function of $r$
$R_n(r)$	eigenfunction
$r_o$	radius of pipe
$r^2$	variable, $1 + (k^2 - 1)(1 - y')$ ; $1 \leq r \leq k$
$S$	inverse heat-flux ratio, $Qa/q_w$
$s$	variable, $r^2 - 1$
$t$	fluid temperature
$U$	dimensionless velocity, $u/u$
$u$	fluid velocity in x-direction
$\bar{u}$	fluid mean velocity
$x$	longitudinal coordinate measured from channel entrance
$x'$	dimensionless longitudinal coordinate, $4(x/a)/\text{Pe}$
$y$	transverse coordinate measured from channel centerline

$y'$	dimensionless transverse coordinate, $y/a$
$z$	transverse coordinate measured from channel wall, $a - y$
$\alpha$	molecular diffusivity for heat, $\kappa/\rho c_p$ ; dummy variable of integration
$\alpha_n^2$	eigenvalue
$\beta$	dummy variable of integration
$\beta_n^2$	eigenvalue
$\epsilon_H$	eddy diffusivity of heat
$\epsilon_M$	eddy diffusivity of momentum
$\eta$	variable, $1 - y'$
$\kappa$	fluid thermal conductivity
$\lambda_n^2$	eigenvalue, $4(m + 1)\alpha_n^2/(k^2 - 1)^{m+2}$
$\lambda_T$	dimensionless diffusivity, $\text{Pr}(\epsilon_H/\nu)$
$\nu$	fluid kinematic viscosity
$\xi$	dummy variable of integration
$\rho$	fluid density
$\tau$	dummy variable of integration
$\varphi$	dummy variable of integration
$\psi$	ratio of eddy diffusivity for heat transfer to that for momentum transfer, $\epsilon_H/\epsilon_M$
$\overline{\psi}$	average value of $\psi$
$\omega_n^2$	eigenvalue, $(m + 1)\beta_n^2$

Subscripts:

b	bulk condition
d	developed region
e	entrance region
i	entrance value
Q	internal heat generation, insulated wall
q	wall heat transfer, no heat generation
w	wall
0	either no heat transfer or no heat generation

## REFERENCES

1. Inman, Robert M. : Heat Transfer to Liquid-Metal Flow in a Round Tube or Flat Duct With Heat Sources in the Fluid Stream. NASA TN D-3473, 1966.
2. Inman, Robert M. : Theoretical Investigation of Turbulent Liquid-Metal Heat Transfer in Channels With Heat Sources in the Fluid. NASA TN D-3692, 1966.
3. Lin, C. C. : Turbulent Flows and Heat Transfer. Princeton University Press, 1959, p. 119.
4. Claiborn, H. C. : Heat Transfer in Noncircular Ducts, Pt. I. Rept. No. ORNL-985, Oak Ridge Nat. Lab., May 14, 1951.
5. Poppendiek, H. F. : Forced Convection Heat Transfer in Thermal Entrance Regions, Pt. I. Rept. No. ORNL-913, Oak Ridge Nat. Lab., Mar. 20, 1951.
6. Poppendiek, H. F. ; and Palmer, L. D. : Forced Convection Heat Transfer in Thermal Entrance Regions, Pt. II. Rept. No. ORNL-914, Oak Ridge Nat. Lab., June 11, 1952.
7. Poppendiek, H. F. : Liquid-Metal Heat Transfer. Heat Transfer Symposium, Engr. Res. Inst., University of Michigan, 1953, pp. 77-100.
8. Hartnett, James P. ; and Irvine, Thomas F., Jr. : Nusselt Values For Estimating Turbulent Liquid Metal Heat Transfer in Noncircular Ducts. AIChE J., vol. 3, no. 3, Sept. 1957, pp. 313-317.
9. Poppendiek, H. F. : Heat Transfer in a Liquid Metal Flowing Turbulently Through a Channel With a Step Function Boundary Temperature. NASA Memo 2-5-59W, 1959.
10. Poppendiek, H. F. : Turbulent Liquid-Metal Heat Transfer in Channels. Nucl. Sci. Eng., vol. 5, no. 6, June 1959, pp. 390-404.
11. Pearson, J. T., Jr. ; and Irvine, T. F., Jr. : An Analog Experiment on Turbulent Liquid Metal Heat Transfer in the Entrance Region of Noncircular Ducts. Fluid Mechanics. Vol. 2, Part I of Developments in Mechanics. Simon Ostrach and R. H. Scanlan, eds., Pergamon Press, 1965, pp. 361-375.
12. Harris, Lawson P. : Hydromagnetic Channel Flows. M.I.T. Tech. Press and John Wiley and Sons, Inc., 1960.
13. Kirko, Igoř M. : Magnetohydrodynamics of Liquid Metals. Consultants Bureau, 1965.
14. Elliott, D. ; Cerini, D. ; and O'Connor, D. : Liquid MHD Power Conversion. Space Programs Summary No. 37-26, vol. IV, Jet Propulsion Lab., California Inst. Tech., Apr. 30, 1964, pp. 124-133.

15. Elliott, David G. : Direct Current Liquid-Metal Magnetohydrodynamic Power Generation. AIAA J., vol. 4, no. 4, Apr. 1966, pp. 627-634.
16. Dwyer, O. E. : Eddy Transport in Liquid-Metal Heat Transfer. AIChE J., vol. 9, no. 2, Mar. 1963, pp. 261-268.
17. Schneider, P. J. : Effect of Axial Fluid Conduction on Heat Transfer in the Entrance Regions of Parallel Plates and Tubes. Trans. ASME, vol. 79, no. 4, May 1957, pp. 765-773.
18. Loeffler, Albert L. , Jr. : Heat Transfer in Fully Developed Flow Between Parallel Plates With Variable Heat Sources. Nucl. Sci. Eng., vol. 2, no. 5, Sept. 1957, pp. 547-567.
19. Dwyer, O. E. : Heat Transfer to Liquid Metals Flowing Turbulently Between Parallel Plates. Nucl. Sci. Eng., vol. 21, no. 1, Jan. 1965, pp. 79-89.
20. Harrison, W. B. ; and Menke, J. R. : Heat Transfer to Liquid Metals Flowing in Asymmetrically Heated Channels. Trans. ASME, vol. 71, no. 7, Oct. 1949, pp. 797-803.

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